Buckling of Open Cylindrical Shells under Combined Compression and Bending Stress

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In this paper, a procedure is developed to overcome the mathematical difficulty in solving the Donnell equations for a curved panel subject to circumferential variation of axial stress. The present paper deals with the small deflection theory where the axial deformation is predominant. The problem of a simply supported open shell subjected to linear variation of axial stress across the cross section, which can be expressed in terms of an infinite series along the circumferential coordinate, is solved in detail. The expressions for the displacements are given in terms of arbitrary undetermined constants, which can be made to satisfy prescribed longitudinal edge conditions. The solutions for displacement in this paper are given as a sum of two parts, the first part representing the contribution due to a constant axial stress and the second part representing the contribution due to the deviation function.

Nomenclature

A_1 to A_4 = arbitrary constants
a_i = coefficients of w series; Eq. (11)
$a_{i,j}$, $\bar{a}_{i,j}$ = coefficients of w series relating to the root $\theta_j, \bar{\theta}_j$, respectively; Eq. (18)
B = half-width of the shell
E = modulus of elasticity
$f_{i,j}$ = see Eqs. (26-30)
K_x = see Eq. (6)
L = length of shell
n = number of terms desired in series expansion of u, v, and w
p_1 to p_6, $\bar{p}_1$ to $\bar{p}_6$ = see Eq. (25b)
p_7 to p_9, $\bar{p}_7$ to $\bar{p}_9$ = see Eq. (31)
q_a, q_b = see Eq. (8) and (9), respectively
R = radius of shell to midsurface
s_1 to s_2, $\bar{s}_1$ to $\bar{s}_2$ = see Eq. (25a)
s_1' to s_2', $\bar{s}_1$ to $\bar{s}_2'$ = see Eq. (25a)
u,v,w = displacement in the x,y,z directions, respectively
x,y,z = coordinates axes of the open shell
Z = see Eq. (6)
α = $m\pi/L$
μ = Poisson's ratio
ψ_0 = half subtended angle; Fig. 1
$\sigma_1,\sigma_2,\sigma_x$ = axial compressive stresses; Fig. 1
σ_b,σ_T = bending stress distribution at bottom and top respectively; Fig. 1
θ_1, θ_2, $\bar{\theta}_1$, $\bar{\theta}_2$, θ_3, θ_4, $\bar{\theta}_3$, $\bar{\theta}_4$ = roots; Eq. (15)
ϕ_1 to ϕ_4 = see Eq. (15)

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Introduction

A REVIEW of the literature on the buckling of thin shells up to 1958 can be found in an article by Fung and Sechler.[1] A summary of the available solutions can be found in the *Handbook of Engineering Mechanics*[2] and the *Handbook for Structural Stability.*[3,4] Most of the literature is concerned with the buckling of a cylindrical tube.[5-11] Classical solutions based on the small deflection theory have been obtained only for curved panels subjected to shear and/or uniform axial compression.[12-14]

Because of the difficulty of solving the differential equations and satisfying the longitudinal edge conditions, solutions could not be obtained for open shells with arbitrary variation of axial stress. In this paper, the preceding difficulty has been overcome, and Donnell's shell equations[15] have been solved for linear axial stress distribution, which arises when a shell is subjected to axial force combined with an end moment. The present paper deals with the small deflection theory, where the axial deformation is predominant.

A general type of deflection function with undetermined coefficients, similar to that used for the buckling of plates for various edge conditions,[16] is assumed. The solution derived here can be made to satisfy prescribed edge boundary conditions. It is shown that the obtained solution reduces to the known case of zero end moments.[14] A numerical example has been solved, for nonzero end moments, to prove the validity of the solution.

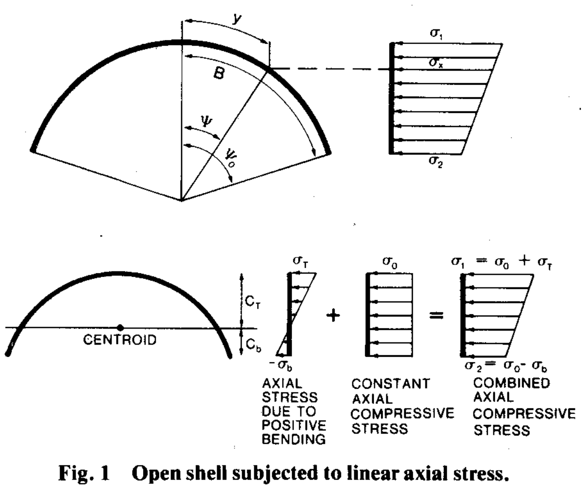

Fig. 1 Open shell subjected to linear axial stress.

Coordinates, Displacements, and Dimensions

The coordinate x is along the longitudinal axis of the shell. The coordinate y is along the circumference of the shell. The x-y surface coincides with the middle surface of the shell. The coordinate z is in the radial direction perpendicular to the x-y surface and is positive outward. The displacements in the positive directions of x, y, and z are referred to as u, v, and w, respectively. The dimensions of the shell are given by length L, thickness t, radius to the middle surface R, half-width B, and half-subtended angle ψ_0.

General Equations

The Donnell's equations for a cylindrical shell subjected to an axial compression stress σ_x are

$$\nabla^8 w + \frac{Et}{R^2 D}\frac{\partial^4 w}{\partial x^4} + \frac{t}{D}\nabla^4\left[\sigma_x \frac{\partial^2 w}{\partial x^2}\right] = 0 \qquad (1)$$

$$\nabla^4 u = -\frac{1}{R}\left[\mu\frac{\partial^3 w}{\partial x^3} - \frac{\partial^3 w}{\partial x \partial y^2}\right] \qquad (2)$$

$$\nabla^4 v = -\frac{1}{R}\left[\left(2+\mu\right)\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right] \qquad (3)$$

where

$$D = Et^3/12(1-\mu^2) \qquad (4)$$

Equation (1) may be rewritten in the form

$$\nabla^8 w + \frac{12z^2}{L^4}\frac{\partial^4 w}{\partial x^4} + \frac{\pi^2}{L^2}\nabla^4\left[K_x \frac{\partial^2 w}{\partial x^2}\right] = 0 \qquad (5)$$

where

$$Z = (L^2/Rt)\ (1-\mu^2)^{1/2} \qquad K_x = (\sigma_x t L^2/D\pi^2) \qquad (6)$$

Equations for Open Shells

The axial stress distribution of a shell subjected to an end moment and constant axial compression is linear, as shown in Fig. 1. Then the value of K_x at any point on the circumference subtending an angle ψ is given by the following equation:

$$K_x = q_a - q_b \cos(y/R) = \sigma_x (tL^2/\pi^2 D) \qquad (7)$$

where

$$q_a = \text{mean axial stress} = \left[\sigma_1 + \frac{\sigma_2 - \sigma_1}{1-\cos\psi_0}\right]\frac{tL^2}{\pi^2 D} \qquad (8)$$

$$q_b = \text{coefficient for deviation function} = \left[\frac{\sigma_2 - \sigma_1}{1-\cos\psi_0}\right]\frac{tL^2}{\pi^2 D} \qquad (9)$$

If an open shell subject to positive end moments has a stress σ_T (compression at top), the corresponding stress at the bottom σ_b can be obtained easily as

$$\sigma_b = -\left(\frac{\sin\alpha/\alpha - \cos\alpha}{1-\sin/\alpha}\right)\sigma_T \qquad (10)$$

In addition to the end moments, if the shell is subject to a constant axial stress of σ_0 (compression), then the total stress at top and bottom can be written as follows: σ_1 = longitudinal stress at top of shell, i.e., y equal to zero $= \sigma_0 + \sigma_T$ (compression positive); σ_2 = longitudinal stress at bottom of shell, i.e., y equal to $B = \sigma_0 + \sigma_b$. The shell is assumed to be simply supported at the ends $x=0$ and $x=L$. The following expression for w, which satisfies these conditions, is assumed to be

$$w = \left[\sum_{i=1}^{n}\frac{a_i}{(i-1)!}\left(\theta\alpha y\right)^{i-1}\right]\sin\alpha x \qquad (11)$$

with

$$a_1 = a_2 = a_3 = a_4 \ldots = a_9 = 1 \qquad (12)$$

where a_i's for $i>9$ shall be determined later, so as to satisfy Eq. (5); θ will be given by the roots of the indicial equation; n is the number of series terms to insure necessary accuracy of the w series; and $\alpha = m\pi/L$, $m = 1,2,3,\ldots$

The restrictions on w coefficients a_1-a_9 are not arbitrary. It will be shown later that the assumed function, in the limiting case of constant axial stress, leads to the exact solution. The procedure indicated here is general enough to be applicable to any stress variation that can be described by a polynomial. In order to demonstrate the feasibility of this method, the stress variation indicated in Eq. (7) will be used.

Expressing $\cos y/R$ in an infinite series form, $\nabla^4[K_x(\delta^2 w/\partial x^2)]$ can be written in terms of $(i-1)$th power of y as

$$\nabla^4\left(K_x\frac{\partial^2 w}{\partial x^2}\right) = -\alpha^2\sum_{i=1}^{n}\nabla^4\left\{\left[q_a\frac{a_i}{(i-1)!} - q_b\sum_{k=1}^{(i+1)/2}\frac{(-1)^{k+1}a_{i+2-2k}}{(i+1-2k)!(2K-2)!}\frac{1}{(\theta\alpha R)^{2k-2}}\right]\left(\theta\alpha y\right)^{i-1}\right\}\sin\alpha x \qquad (13)$$

where the terminal value of k is the integer part of the quotient $(i+1)/2$. Substituting Eqs. (11) and (13) into Eq. (5), the general relationship for the coefficients of $(i-1)$th power of y can be established as follows:

$$\theta^8 a_{i+8} - 4\theta^6 a_{i+6} + \left(6 - \frac{q_a}{m^2}\right)\theta^4 a_{i+4} + \left(\frac{2q_a}{m^2} - 4\right)\theta^2 a_{i+2} + \left(1 + \frac{12z^2}{(m\pi)^4} - \frac{q_a}{m^2}\right)a_i$$
$$+ \frac{q_b}{m^2}\sum_{k=1}^{(i+1)/2}\frac{(-1)^{k+1}a_{i+2-2k}}{(i+1-2k)!\,(2k-2)!}\frac{(i-1)!}{(\theta\alpha R)^{2k-2}} - \frac{2q_b}{m^2}\theta^2\sum_{k=1}^{(i+3)/2}\frac{(-1)^{k+1}a_{i+4-2k}}{(i+3-2k)!\,(2k-2)!}\frac{(i+1)!}{(\theta\alpha R)^{2k-2}}$$
$$+ \frac{q_b}{m^2}\theta^4\sum_{k=1}^{(i+5)/2}\frac{(-1)^{k+1}a_{i+6-2k}}{(i+5-2k)!\,(2k-2)!}\frac{(i+3)!}{(\theta\alpha R)^{2k-2}} = 0 \qquad (14)$$

Solution for θ

For $i=1$, with the first nine constants a_1-a_9 set equal to unity, Eq. (14) becomes a biquadratic in θ^2, which can be solved explicitly for four roots; this results in two sets of complex conjugates. In turn, this can be solved explicitly for θ, which will result in eight roots, listed as follows:

$$\theta_1 = \phi_1 + \sqrt{-1}\,\phi_2 = -\theta_3 \qquad \bar{\theta}_1 = \phi_1 - \sqrt{-1}\,\phi_2 = -\bar{\theta}_3$$
$$\theta_2 = \phi_3 + \sqrt{-1}\,\phi_4 = -\theta_4 \qquad \bar{\theta}_2 = \phi_3 - \sqrt{-1}\,\phi_4 = -\bar{\theta}_4 \qquad (15)$$

It must be pointed out that the shells subjected to axial stresses higher than the classical buckling stress

$$\sigma_{cl} = Et/[\sqrt{3}\,R(1-\mu^2)^{1/2}] \qquad (16)$$

may have roots different from those given in Eq. (15). Since the primary interest is in shells having stress distribution less than or equal to σ_{cl}, this paper is concerned with the solutions of such a class of problems.

Solution of w in Terms of Arbitrary Constants

Solution of Eq. (14) for a_{i+8}, for values of i greater than 1, yields the following recursive relationship:

$$a_{i+8}=a_i-\frac{4}{\theta^2}\left(a_i-a_{i+6}\right)+\left(6-\frac{q_a}{m^2}\right)\frac{1}{\theta^4}\left(a_i-a_{i+4}\right)$$
$$+\left(\frac{2q_a}{m^2}-4\right)\frac{1}{\theta^6}\left(a_i-a_{i+2}\right)$$
$$+\frac{q_b}{m^2}\left[\frac{1}{\theta^4}-2\left(1+\frac{3}{(\alpha R)^2}\right)\frac{1}{\theta^6}+\left(1+\frac{1}{(\alpha R)^2}\right)^2\times\frac{1}{\theta^8}\right]a_i$$
$$-\frac{q_b}{m^2}\sum_{k=1}^{(i+1)/2}\frac{(-1)^{k+1}a_{i+2-2k}}{(i+1-2k)!\,(2k-2)!}\frac{(i-1)!}{(\theta\alpha R)^{2k-2}}x\frac{1}{\theta^8}$$
$$+\frac{2q_b}{m^2}\sum_{k=1}^{(i+3)/2}\frac{(-1)^{k+1}a_{i+4-2k}}{(i+3-2k)!\,(2k-2)!}\frac{(i+1)!}{(\theta\alpha R)^{2k-2}}x\frac{1}{\theta^6}$$
$$-\frac{q_b}{m^2}\sum_{k=1}^{(i+5)/2}\frac{(-1)^{k+1}a_{i+6-2k}}{(i+5-2k)!\,(2k-2)!}\frac{(i+3)!}{(\theta\alpha R)^{2k-2}}x\frac{1}{\theta^4}\qquad(17)$$

It should be noted that, since the preceding equation contains only even powers of θ, the coefficient a_{i+8} will have the same value for $+\theta$ as well as for $-\theta$.

The solution of w is obtained by substituting Eq. (15) in Eq. (11). It will involve eight different series, corresponding to eight roots of θ. For a shell with symmetric boundary conditions, it is desirable to separate the symmetrical and antisymmetrical deflection pattern. These types of formulations will aid in solving specific stability problems, such as buckling of open shells.[14] The symmetric deflection condition $w(y)=+w(-y)$ requires that all of the odd powers of y must vanish, and the antisymmetric deflection condition $w(y)=-w(-y)$ requires that all of the even powers of y must vanish. By use of the previous requirements, the symmetrical and antisymmetrical solutions for w can be written in terms of four new constants:

$$\left.\begin{matrix}w_S\\w_A\end{matrix}\right\}=\sum_{i=1}^{n}\left[\frac{A_1(a_{i,1}\theta_1^{i-1}-\bar a_{i,1}\bar\theta_1^{i-1})}{\sqrt{-1}+A_2(a_{i,1}\theta_1^{i-1}+\bar a_{i,1}\bar\theta_1^{i-1})}\right.$$
$$+A_3\left(a_{i,2}\theta_2^{i-1}-\bar a_{i,2}\bar\theta_2^{i-1}/\sqrt{-1}\right)$$
$$\left.+A_4\left(a_{i,2}\theta_2^{i-1}+\bar a_{i,2}\bar\theta_2^{i-1}\right)\right]\frac{(\alpha y)^{i-1}}{(i-1)!}\sin\alpha x,\quad i=\begin{matrix}1,3,5,7,\ldots\\2,4,6,8,\ldots\end{matrix}\qquad(18)$$

Expressions for u, v, and w

Any axial stress distribution can be written as a sum of two parts, the first part corresponding to a constant mean axial stress[14] and the second part corresponding to a variation from the constant axial stress. In order to separate these parts, the complex coefficients $a_{i,j}$, $\bar a_{i,j}$ and the complex roots θ_j and $\bar\theta_j$ are written in the following forms:

$$\bar a_{i,j}=1+\bar\epsilon_{i,j}=1+\beta_{i,j}(\cos\delta_{i,j}-\sqrt{-1}\,\sin\delta_{i,j})$$
$$\theta_j=r_j(\cos\psi_j+\sqrt{-1}\,\sin\psi_j)\qquad\bar\theta_j=r_j(\cos\psi_j-\sqrt{-1}\,\sin\psi_j)$$

where

$$\beta_{i,j}=|\epsilon_{i,j}|=|\bar\epsilon_{i,j}|\qquad\xi_j=|\theta_j|=|\bar\theta_j|$$

and

$$\delta_{i,j}=\text{argument of }\epsilon_{i,j}\qquad\psi_j=\text{argument of }\theta_j\qquad(19)$$

using the above expressions it is easy to derive the following relationship for w:

$$\left.\begin{matrix}w_S\\w_A\end{matrix}\right\}=\left\{A_1\begin{matrix}\sinh\\\cosh\end{matrix}\,\phi_1\alpha y\,\sin\phi_2\alpha y+A_2\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_1\alpha y\,\cos\phi_2\alpha y\right.$$
$$\left.+A_3\begin{matrix}\sinh\\\cosh\end{matrix}\,\phi_3\alpha y\,\sin\phi_4\alpha y+A_4\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_3\alpha y\,\cos\phi_4\alpha y\right\}\sin\alpha x$$
$$+\left\{A_1\sum_i\left[\beta_{i,1}\sin\left((i-1)\psi_1-\delta_{i,1}\right)\right]\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right.$$
$$+A_2\sum_i\beta_{i,1}\cos\left(\delta_{i,1}-(i-1)\psi_1\right)\Big]\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}$$
$$+A_3\sum_i\beta_{i,2}\sin\left((i-1)\psi_2-\delta_{i,2}\right)\Big]\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}$$
$$\left.+A_4\sum_i\left[\beta_{i,2}\cos\left(\delta_{i,2}-(i-1)\psi_2\right)\right]\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right\}$$
$$\sin\alpha\,x\qquad i=\begin{matrix}11,13,15,\ldots\\10,12,14,\ldots\end{matrix}\qquad(20)$$

The preceding series of w can now be used to obtain the expressions for u and v by making use of the relationships given by Eq. (2) and (3), respectively. The expressions thus obtained are

$$\left.\begin{matrix}u_s\\u_a\end{matrix}\right\}=\left\{A_1\left[s_1\begin{matrix}\sinh\\\cosh\end{matrix}\,\phi_1\alpha y\,\sin\phi_2\alpha y+s_2\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_1\alpha y\cos\phi_2\alpha y\right]\right.$$
$$\frac{1}{p_1^2+p_2^2}\quad+A_1\sum_i\gamma_{i,1}\sin\psi_{i,1}^*\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}$$
$$+A_2\left[s_1\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_1\alpha y\,\cos\phi_2\alpha y-s_2\begin{matrix}\sinh\\\cosh\end{matrix}\,\phi_1\alpha y\,\sin\phi_2\alpha y\right]\frac{1}{p_1^2+p_2^2}$$
$$\times\frac{1}{p_1^2+p_2^2}+A_2\sum_i\gamma_{i,1}\cos\psi_{i,1}^*\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}$$
$$+A_3\left[\bar s_1\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_3\alpha y\,\sin\phi_4\alpha y+\bar s_2\begin{matrix}\sinh\\\cosh\end{matrix}\,\phi_3\alpha y\,\cos\phi_4\alpha y\right.$$
$$+A_3\sum_i\gamma_{i,2}\sin\psi_{i,2}^*\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}$$
$$+A_4\left[\bar s_1\begin{matrix}\cosh\\\sinh\end{matrix}\,\phi_3\alpha y\,\cos\phi_4\alpha y-\bar s_2\begin{matrix}\sinh\\\cosh\end{matrix}\phi_3\alpha y\,\cos\phi_4\alpha y\right]$$
$$\frac{1}{\bar p_1^2+\bar p_2^2}\quad+A_4\sum_i\gamma_{i,2}\cos\psi_{i,2}^*\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right\}\frac{\cos\alpha\,x}{\alpha R}\qquad(21)$$

$$\gamma_{i,j}=\frac{1}{\alpha R}\beta_{i,j}\{[\mu(\alpha y)^4+(i-1)(i-2)(\alpha y)^2]$$
$$/[(\alpha y)^4-2(i-1)(i-2)(\alpha y)^2$$
$$+(i-1)(i-2)(i-3)(i-4)]\}\qquad(22a)$$

$$\psi_{i,j}^*=(i-1)\psi_j-\delta_{i,j}\qquad i=\begin{matrix}11,13,15,\ldots\\10,12,14,\ldots\end{matrix}$$

$$j=1,2\qquad(22b)$$

and

$$
\left.\begin{matrix} v_S \\ v_A \end{matrix}\right\} = \Bigg\{ A_1\left[s_1' \,{}^{\cosh}_{\sinh}\, \phi_1\alpha y \sin\phi_2\alpha y + s_2' \,{}^{\sinh}_{\cosh}\, \phi_1\alpha y \cos\phi_2\alpha y \right]
$$

$$
\times \frac{1}{p_1^2+p_2^2} + A_1 \sum_i \gamma'_{i,1}\sin\psi^*_{i,1}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}
$$

$$
+A_2\left[s_1' \,{}^{\sinh}_{\cosh}\, \phi_1\alpha y \cos\phi_2\alpha y - s_2' \,{}^{\cosh}_{\sinh}\, \phi_1\alpha y \sin\phi_2\alpha y \right]
$$

$$
\times \frac{1}{p_1^2+p_2^2} + A_2 \sum_i \gamma'_{i,1}\cos\psi^*_{i,1}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}
$$

$$
+A_3\left[\bar{s}_1' \,{}^{\cosh}_{\sinh}\, \phi_3\alpha y \sin\phi_4\alpha y + \bar{s}_2' \,{}^{\sinh}_{\cosh}\, \phi_3\alpha y \cos\phi_4\alpha y \right]
$$

$$
+ A_4 \sum_i \gamma'_{i,2}\cos\psi^*_{i,2}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\Bigg\} \frac{\sin\alpha x}{\alpha R} \tag{23}
$$

$$
\gamma'_{i,j} = \frac{1}{\alpha R}\beta_{i,j}\frac{(2+\mu)(i-1)(\alpha y)^3-(i-1)(i-2)(i-3)(\alpha y)}{(\alpha y)^4-2(i-1)(i-2)(\alpha y)^2+(i-1)(i-2)(i-3)(i-4)} \qquad i = {}^{10,12,14,\ldots}_{11,13,15,\ldots} \qquad j=1,2 \tag{24}
$$

in which

$$
\left.\begin{aligned}
s_1 &= p_1p_3 - p_2p_4 & s_2 &= p_1p_4 + p_2p_3 \\
\bar{s}_1 &= \bar{p}_1\bar{p}_3 - \bar{p}_2\bar{p}_4 & \bar{s}_2 &= \bar{p}_1\bar{p}_4 + \bar{p}_2\bar{p}_3 \\
s_1' &= p_1p_5 - p_2p_6 & s_2' &= p_1p_6 + p_2p_5 \\
\bar{s}_1' &= \bar{p}_1\bar{p}_5 - \bar{p}_2\bar{p}_6 & \bar{s}_2' &= \bar{p}_1\bar{p}_6 + \bar{p}_2\bar{p}_5
\end{aligned}\right\} \cdots\cdots \tag{25a}
$$

and

$$
\begin{aligned}
p_1 &= 1-2(\phi_1^2-\phi_2^2)+(\phi_1^4-6\phi_1^2\phi_2^2+\phi_2^4) = 2p^2q \\
\bar{p}_1 &= 1-2(\phi_3^2-\phi_4^2)+(\phi_3^4-6\phi_3^2\phi_4^2+\phi_4^4) = 2p^2q = p_1 \\
p_2 &= 4\phi_1\phi_2(1-\phi_1^2+\phi_2^2) = -2p^2\sqrt{-q^2+r^2} \\
\bar{p}_2 &= 4\phi_3\phi_4(1-\phi_3^2+\phi_4^2) = 2p^2\sqrt{-q^2+r^2} = -p_2 \\
p_3 &= \mu+(\phi_1^2-\phi_2^2) = (1+\mu)+p\sqrt{q+r} \\
\bar{p}_3 &= \mu+(\phi_3^2-\phi_4^2) = (1+\mu)-p\sqrt{q+r} \\
p_4 &= 2\phi_1\phi_2 = p\sqrt{-q+r} \qquad \bar{p}_4 = 2\phi_3\phi_4 = p\sqrt{-q+r} = p_4 \\
p_5 &= \phi_1[(2+\mu)-(\phi_1^2-3\phi_2^2)] \\
\bar{p}_5 &= \phi_3[(2+\mu)-(\phi_3^2-3\phi_4^2)] \\
p_6 &= \phi_2[(2+\mu)-(3\phi_1^2-\phi_2^2)] \\
p_6 &= \phi_4[(2+\mu)-(3\phi_3^2-\phi_4^2)]
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 &= \frac{1}{\alpha R}\frac{1}{4p^4r^2} = \frac{1}{\alpha R}\frac{1}{p_1^2+p_2^2} = \frac{1}{\alpha R}\frac{1}{\bar{p}_1^2+\bar{p}_2^2} = \rho_2 \\
4p^4r^2 &= p_1^2+p_2^2 = \bar{p}_1^2+\bar{p}_2^2 = \frac{12(1-\mu^2)}{(m\pi)^4}\left(\frac{L}{t}\right)^2\left(\frac{L}{R}\right)^2
\end{aligned} \tag{25b}
$$

Special Case of Constant Axial Stress ($\sigma_T = 0$)

In order to substantiate the validity of the derived solutions for u, v, and w, the limiting case in which the axial stress is constant is examined. In the case of constant axial stress distribution,

$$
\sigma_T = 0 \quad \sigma_2 = \sigma_1 \quad q_a = \sigma_1(tL^2/\pi^2D) \quad q_b = 0
$$

This implies that the terms involving q_b will vanish, and all of the coefficients a_i become unity, thus reducing the solutions of u, v, and w to the closed-form solutions given in Ref. 14.

Numerical Example ($\sigma_T \neq 0$)

A numerical example has been solved to determine the buckling stress of a simply supported open shell for the case when linear axial variation of axial stress σ_T is equal to $\pm 10^{-5}$th of σ_{cl}.

Equations for N_y, N_{xy}, N_y and V_y

Equations for M_y, N_{xy}, N_y, and V_y can be written in the following form, using the already derived expressions for u, v, and w:

$$
\left.\begin{matrix}(M_y)_S\\(M_y)_A\end{matrix}\right\} = \Bigg\{ A_1 e^{\phi_1\alpha y} \begin{matrix}[f_{11}(y)]_S\\ [f_{11}(y)]_A\end{matrix} + A_2 e^{\phi_1\alpha y} \begin{matrix}[f_{12}(y)]_S\\ [f_{12}(y)]_A\end{matrix}
+ A_3 e^{\phi_3\alpha y} \begin{matrix}[f_{13}(y)]_S\\ [f_{13}(y)]_A\end{matrix} + A_4 e^{\phi_3\alpha y} \begin{matrix}[f_{14}(y)]_S\\ [f_{14}(y)]_A\end{matrix}\Bigg\} \frac{\rho_2}{2}\sin\alpha x \ldots \tag{26a}
$$

$$
\left.\begin{matrix}(N_{xy})_S\\(N_{xy})_A\end{matrix}\right\} = \Bigg\{\left[A_1 e^{\phi_1\alpha y} \begin{matrix}[f_{21}(y)]_S\\ [f_{21}(y)]_A\end{matrix} + A_2 e^{\phi_1\alpha y} \begin{matrix}[f_{22}(y)]_S\\ [f_{22}(y)]_A\end{matrix}\right]\frac{\rho_3}{2}
+\left[A_3 e^{\phi_3\alpha y} \begin{matrix}[f_{23}(y)]_S\\ [f_{23}(y)]_A\end{matrix} + A_4 e^{\phi_3\alpha y} \begin{matrix}[f_{24}(y)]_S\\ [f_{24}(y)]_A\end{matrix}\right]\frac{\bar{\rho}_3}{2}\Bigg\}\cos\alpha x \ldots \tag{26b}
$$

$$
\left.\begin{matrix}(N_y)_S\\(N_y)_A\end{matrix}\right\} = \Bigg\{\left[A_1 e^{\phi_1\alpha y} \begin{matrix}[f_{31}(y)]_S\\ [f_{31}(y)]_A\end{matrix} + A_2 e^{\phi_1\alpha y} \begin{matrix}[f_{32}(y)]_S\\ [f_{32}(y)]_A\end{matrix}\right]\frac{\rho_3}{2}
+\left[A_3 e^{\phi_3\alpha y} \begin{matrix}[f_{33}(y)]_S\\ [f_{33}(y)]_A\end{matrix} + A_4 e^{\phi_3\alpha y} \begin{matrix}[f_{34}(y)]_S\\ [f_{34}(y)]_A\end{matrix}\right]\frac{\bar{\rho}_3}{2}\Bigg\}\sin\alpha x \ldots \tag{26c}
$$

$$
\left.\begin{matrix}(V_y)_S\\(V_y)_A\end{matrix}\right\} = \Bigg\{ A_1 e^{\phi_1\alpha y} \begin{matrix}[f_{41}(y)]_S\\ [f_{41}(y)]_A\end{matrix} + A_2 e^{\phi_1\alpha y} \begin{matrix}[f_{42}(y)]_S\\ [f_{42}(y)]_A\end{matrix}
+ A_3 e^{\phi_3\alpha y} \begin{matrix}[f_{43}(y)]_S\\ [f_{43}(y)]_A\end{matrix} + A_4 e^{\phi_3\alpha y} \begin{matrix}[f_{44}(y)]_S\\ [f_{44}(y)]_A\end{matrix}\Bigg\} \frac{\alpha\rho_2}{2}\sin\alpha x \ldots \tag{26d}
$$

in which

$$
\begin{aligned}
\left.\begin{matrix}[f_{11}(y)]_S\\ [f_{11}(y)]_A\end{matrix}\right\} &= p_7(1\pm e^{-2\phi_1\alpha y})\sin\phi_2\alpha y \\
&\quad + p_4(1\pm e^{-2\phi_1\alpha y})\cos\phi_2\alpha y \\
&\quad + 2\sum_i \beta_{i,1}\sin\psi^*_{i,1}\left[\frac{(i-1)(i-2)}{(\alpha y)^2}-\mu\right]\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right]
\end{aligned}
$$

$$
\begin{aligned}
\left.\begin{matrix}[f_{12}(y)]_S\\ [f_{12}(y)]_A\end{matrix}\right\} &= p_7(1\pm e^{-2\phi_1\alpha y})\cos\phi_2\alpha y \\
&\quad - p_4(1\mp e^{-2\alpha y})\sin\phi_2\alpha y \\
&\quad + 2\sum_i \beta_{i,1}\cos\psi^*_{i,1}\left[\frac{(i-1)(i-2)}{(\alpha y)^2}-\mu\right]\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right]
\end{aligned}
$$

$$\left.\begin{matrix}[f_{13}(y)]_S\\ [f_{13}(y)]_A\end{matrix}\right\} = \bar{p}_7(1\mp e^{-2\phi_3\alpha y})\sin\phi_4\alpha y + \bar{p}_4(1\pm e^{-2\phi_3\alpha y})\cos\phi_4\alpha y + 2\sum_i \beta_{i,2}\sin\psi^*_{i,2}\left[\frac{(i-1)(i-2)}{(\alpha y)^2}-\mu\right]\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right]$$

$$\left.\begin{matrix}[f_{14}(y)]_S\\ [f_{14}(y)]_A\end{matrix}\right\} = \bar{p}_7(1\pm e^{-2\phi_3\alpha y})\cos\phi_4\alpha y - \bar{p}_4(1\mp e^{-2\phi_3\alpha y})\sin\phi_4\alpha y + 2\sum_i \beta_{i,2}\cos\psi^*_{i,2}\left[\frac{(i-1)(i-2)}{(\alpha y)^2}-\mu\right]\left[e^{-\phi_3\alpha y}\frac{[\xi_2\alpha y)^{i-1}}{(i-1)!}\right]$$

$$\left.\begin{matrix}[f_{21}(y)]_S\\ [f_{21}(y)]_A\end{matrix}\right\} = [q_1(1\pm e^{-2\phi_1\alpha y})\sin\phi_2\alpha y + q_2(1\mp e^{-2\phi_1\alpha y})\cos\phi_2\alpha y]\frac{1}{p_1^2+p_2^2} + \frac{1}{1+\mu}\sum_i \gamma_{i,1}\sin\psi^*_{i,1}\left(\frac{i-1}{\alpha y}\right)\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right] + \frac{1+\mu}{1}\sum_j \gamma'_{j,1}\sin\psi^*_{j,1}\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{22}(y)]_S\\ [f_{22}(y)]_A\end{matrix}\right\} = [q_1(1\pm e^{-2\phi_1\alpha y})\cos\phi_2\alpha y - q_2(1\pm e^{-2\phi_1\alpha y})\sin\phi_2\alpha y]\frac{1}{p_1^2+p_2^2} + \frac{1}{1+\mu}\sum_i \gamma_{i,1}\cos\psi^*_{i,1}\left(\frac{i-1}{\alpha y}\right)\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right] + \frac{1}{1+\mu}\sum_j \gamma'_{j,1}\cos\psi^*_{j,1}\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{23}(y)]_S\\ [f_{23}(y)]_A\end{matrix}\right\} = [q_1(1\pm e^{-2\phi_3\alpha y})\sin\phi_4\alpha y + q_2(1\mp e^{-2\phi_3\alpha y})\cos\phi_4\alpha y]\frac{1}{\bar{p}_1^2+\bar{p}_2^2} + \frac{1}{1+\mu}\sum_i \gamma_{i,2}\sin\psi^*_{i,2}\left(\frac{i-1}{\alpha y}\right)\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right] + \frac{1}{1+\mu}\sum_j \gamma'_{j,2}\sin\psi^*_{j,2}\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{24}(y)]_S\\ [f_{24}(y)]_A\end{matrix}\right\} = [\bar{q}_1(1\mp e^{-2\phi_3\alpha y})\cos\phi_4\alpha y - q_2(1\pm e^{-2\phi_3\alpha y})\sin\phi_4\alpha y]\frac{1}{\bar{p}_1^2+\bar{p}_2^2} + \frac{1}{1+\mu}\sum_i \gamma_{i,2}\cos\psi^*_{i,2}\left(\frac{i-1}{\alpha y}\right)\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right] + \frac{1}{1+\mu}\sum_j \gamma'_{j,2}\cos\psi^*_{j,2}\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{j-1}}{(j-1)!}\right] \quad (28)$$

$$\left.\begin{matrix}[f_{31}(y)]_S\\ [f_{31}(y)]_A\end{matrix}\right\} = [p_1(1\mp e^{-2\phi_1\alpha y})\sin\phi_2\alpha y + p_2(1\pm e^{-2\phi_1\alpha y})\cos\phi_2\alpha y]\frac{1}{p_1^2+p_2^2} + \frac{2}{1-\mu^2}\sum_i(\beta_{i,1}-\mu\gamma_{i,1})\sin\psi^*_{i,1}\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right] + \frac{2}{1-\mu^2}\sum_j \gamma'_{j,1}\sin\psi^*_{j,1}\left(\frac{j-1}{\alpha y}\right)\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{32}(y)]_S\\ [f_{32}(y)]_A\end{matrix}\right\} = [p_1(1\pm e^{-2\phi_1\alpha y})\cos\phi_2\alpha y - p_2(1\mp e^{-2\phi_1\alpha y})\sin\phi_2\alpha y]\frac{1}{p_1^2+p_2^2} + \frac{2}{1-\mu^2}\sum_i(\beta_{i,1}-\mu\gamma_{i,1})\cos\psi^*_{i,1}\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right] + \frac{2}{1-\mu^2}\sum_j \gamma'_{j,1}\left(\frac{j-1}{\alpha y}\right)\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{33}(y)]_S\\ [f_{33}(y)]_A\end{matrix}\right\} = [\bar{p}_1(1\mp e^{-2\phi_3\alpha y})\sin\phi_4\alpha y + \bar{p}_2(1\pm e^{-2\phi_3\alpha y})\cos\phi_4\alpha y]\frac{1}{\bar{p}_1^2+\bar{p}_2^2} + \frac{2}{1-\mu^2}\sum_i(\beta_{i,1}-\mu\gamma_{i,1})\sin\psi^*_{i,1}\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right] + \frac{2}{1-\mu^2}\sum_j \gamma'_{j,1}\sin\psi^*_{j,1}\left(\frac{j-1}{\alpha y}\right)\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{j-1}}{(j-1)!}\right]$$

$$\left.\begin{matrix}[f_{34}(y)]_S\\ [f_{34}(y)]_A\end{matrix}\right\} = [\bar{p}_1(1\pm e^{-2\phi_3\alpha y})\cos\phi_4\alpha y - \bar{p}_2(1\mp e^{-2\phi_3\alpha y})\sin\phi_4\alpha y]\frac{1}{\bar{p}_1^2+\bar{p}_2^2} + \frac{2}{1-\mu^2}\sum_i(\beta_{i,2}-\mu\gamma_{i,2})\cos\psi^*_{i,2}\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right] + \frac{2}{1-\mu^2}\sum_j \gamma'_{j,2}\cos\psi^*_{j,2}\left(\frac{j-1}{\alpha y}\right)\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{j-1}}{(j-1)!}\right] \quad (29)$$

$$\left.\begin{matrix}[f_{41}(y)]_S\\ [f_{41}(y)]_A\end{matrix}\right\} = p_8(1\pm e^{-2\phi_1\alpha y})\sin\phi_2\alpha y + p_9(1\mp e^{-2\phi_1\alpha y})\cos\phi_2\alpha y + 2\sum_i \beta_{i,1}\sin\psi^*_{i,1}\left[\frac{(i-1)(i-2)(i-3)}{(\alpha y)^3}-(2-\mu)\left(\frac{i-1}{\alpha y}\right)\right]\times\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right]$$

$$\left.\begin{matrix}[f_{42}(y)]_S\\ [f_{42}(y)]_A\end{matrix}\right\} = p_8(1\mp e^{-2\phi_1\alpha y})\cos\phi_2\alpha y - p_9(1\pm e^{-2\phi_1\alpha y})\sin\phi_2\alpha y + 2\sum_i \beta_{i,1}\cos\psi^*_{i,1}\left[\frac{(i-1)(i-2)(i-3)}{(\alpha y)^3}-(2-\mu)\left(\frac{i-1}{\alpha y}\right)\right]\times\left[e^{-\phi_1\alpha y}\frac{(\xi_1\alpha y)^{i-1}}{(i-1)!}\right]$$

$$\begin{array}{l}[f_{43}(y)]_S\\ [f_{43}(y)]_A\end{array} = \bar{p}_8(1 \pm e^{-2\phi_3\alpha y})\sin\phi_4\alpha y$$
$$+\bar{p}_9(1 \mp e^{-2\phi_3\alpha y})\cos\phi_4\alpha y$$
$$+2\sum_i \beta_{1,2}\sin\psi^*_{i,2}\left[\frac{(i-1)(i-2)(i-3)}{(\alpha y)^3} - (2-\mu)\left(\frac{i-1}{\alpha y}\right)\right]$$
$$\times\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right]$$

$$\left.\begin{array}{l}[f_{44}(y)]_S\\ [f_{44}(y)]_A\end{array}\right\} = \bar{p}_8(1 \mp e^{-2\phi_3\alpha y})\cos\phi_4\alpha y$$
$$-\bar{p}_9(1 \pm e^{-2\phi_3\alpha y})\sin\phi_4\alpha y$$
$$+2\sum_i \beta_{i,2}\cos\psi^*_{i,2}\left[\frac{(i-1)(i-2)(i-3)}{(\alpha y)^3} - (2-\mu)\left(\frac{i-1}{\alpha y}\right)\right]$$
$$\times\left[e^{-\phi_3\alpha y}\frac{(\xi_2\alpha y)^{i-1}}{(i-1)!}\right] \qquad \begin{array}{l} i=11,13,15,\ldots\\ \quad 10,12,14,\ldots\\ j=11,13,15,\ldots\\ \quad 10,12,14,\ldots\end{array} \tag{30}$$

in which

$$p_7=\phi_1^2-\phi_2^2-\mu \qquad \bar{p}_7=\phi_3^2-\phi_4^2-\mu$$
$$p_8=\phi_1(\phi_1^2-3\phi_2^2-2+\mu) \qquad \bar{p}_8=\phi_3(\phi_3^2-3\phi_4^2-2+\mu)$$
$$p_9=\phi_2(3\phi_1^2-\phi_2^2-2+\mu) \qquad \bar{p}_9=\phi_4(3\phi_3^2-\phi_4^2-2+\mu)$$
$$q_1=p_1\phi_1-p_2\phi_2 \qquad \bar{q}_1=\bar{p}_1\phi_3-\bar{p}_2\phi_4$$
$$q_2=p_1\phi_2+p_2\phi_1 \qquad \bar{q}_2=\bar{p}_1\phi_4+\bar{p}_2\phi_3$$
$$\rho_2=-\frac{\alpha^2Et^3}{12(1-\mu^2)} \qquad \rho_3=\frac{Et}{R}\frac{1}{4p^4r^2}=\frac{Et}{R}\frac{1}{p_1^2+p_2^2}$$
$$\bar{\rho}_3=\frac{Et}{R}\frac{1}{\bar{p}_1^2+\bar{p}_2^2} \tag{31}$$

Determination of Critical Buckling Stress

For symmetric or antisymmetric buckling, the following boundary conditions must be satisfied along the free longitudinal edge:

$$M_y=N_{xy}=N_y=V_y=0 \quad \text{at} \quad y=+B$$

where B is the half-width of shell. For $y=+B$, the equations for buckling stress may be written as

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22} & c_{23} & c_{24}\\ c_{31} & c_{32} & c_{33} & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}\end{bmatrix}\begin{bmatrix} A_1 & e^{\phi_1\alpha B}\\ A_2 & e^{\phi_1\alpha B}\\ A_3 & e^{\phi_2\alpha B}\\ A_4 & e^{\phi_2\alpha B}\end{bmatrix}=0 \tag{32}$$

where $c_{ij}=f_{ij}(B)$. Since the compressive stress is involved implicitly in all of the elements of the matrix C, the critical value of the constant axial compressive stress for a given bending stress distribution is obtained numerically, which will make $|C|$ equal to zero.

Buckling Stress Curves

The nondimensional critical constant axial stress values for the prescribed bending stress distribution of σ_{cr}/E (or axial compressive strain) vs L/R ratios are plotted in Fig. 2, for a half-substended angle $\psi_0=45°$ and $t/R=0.01$, for the following three cases: 1) σ_{cr}/E for zero bending stress; 2) σ_{cr}/E for combined axial compressive stress, with a positive bending moment, which causes a compressive stress equal to $\sigma_{cl}10^{-5}$ at the crown; that is, $\psi=0°$; and 3) σ_{cr}/E for combined axial compressive stress, with a negative bending moment, which causes a tensile stress equal to $\sigma_{cl}\times 10^{-5}$ at the crown; that is, $\psi=0°$.

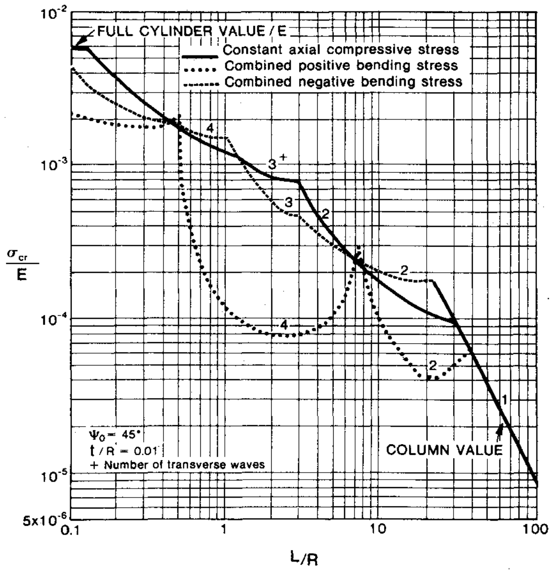

Fig. 2 Curves of σ_{cr}/E vs L/R.

Discussion of Results and Conclusions

The first case of constant axial stress, without any bending, was examined in order to verify the developed theory and numerical computations. This case yielded results identical to those given in Ref. 14. The second and third cases were examined to compute the axial stress; that is, σ_{cr} necessary to cause buckling for a shell with a prescribed boundary stress distribution, which arises because of initial eccentricity or dead and live loads on a simply supported shell.

The curves σ_{cr}/E plotted against L/R values in Fig. 2 indicate that the theoretical and numerical procedure developed here yields the closed-form solutions already obtained.[14] As expected, a shell subjected to positive bending or bending causing compression at the top and tension at the edges can be either as strong, or only 1/10th as strong, as an open shell subjected to constant axial stress, depending on the values of L/R ratios. However, in very narrow regions, the curve peaks slightly higher than the constant axial stress case. This type of stress distribution forces the shell to buckle in an antisymmetric transverse mode. It can be seen in Fig. 2 that the shells subjected to negative bending or bending causing tension at the top and compression at the edges are definitely stronger than the shells subjected to positive bending stress. It is interesting to note that the shell is either slightly stronger or weaker than a shell subjected to a constant axial stress, depending on the range of values of L/R, undergoing the same mode of buckling, either symmetric or antisymmetric. It can be observed from the curves in Fig. 2 that all three cases appear to converge to almost the same values in narrow regions near $L/R=0.5$ and 7.5.

The study of shells with free longitudinal edges subjected to linear axial compression caused by the combination of constant axial stresses and bending stresses may be regarded as an initial step in studying shells with arbitrary circumferential variation of axial stress by perturbation techniques. The results obtained in this study may be used as an upper bound for studying similar shells by large deflection theory. It is hoped that the technique developed here could be adopted successfully for studying plates and shells with different edge and boundary conditions.

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